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SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 48 (2004) 725–729
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

An Intersection Theorem for Multivalued Maps and Applications

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(Received December 2003; accepted March 2004)

Abstract—A new intersection theorem for multivalued maps is obtained. This new theorem requires the maps involved to satisfy a weaker compactness condition and generalizes known results. Applications of this new theorem are given to the existence of maximal and greatest elements for strict and weak relations and to minimax inequalities.

Keywords—*KKM* maps, Intersection theorem, Maximal elements, Minimax inequality

1. INTRODUCTION

Intersection theorems for multivalued maps have applications to existence problems arising, for example, in mathematical economics and optimization.

Intersection theorems for a multivalued map T defined in a Hausdorff topological vector space are often obtained by imposing suitable closeness and compactness conditions on the family $\{T(x) : x \in X\}$. Fan [1] first considered the class of KKM-maps with closed values and assumed that one of the closed values is compact. In [2], Fan showed that the intersection result in [1] remains valid under the weak condition: $\bigcap_{x\in X_0} T(x) \subset D$, where D is compact. Tian [3] improved the closeness conditions of Fan's result in [2] by considering the class of transfer-closed KKM-maps and assumed that $\bar{T}$ satisfies the compactness condition. However, there are transfer-closed maps whose closures do not satisfy the compactness condition, for any subset X_0 (see Section 2, Example 2.1).

In this paper, we improve the intersection results in [2,3], by relaxing the compactness condition. We impose a weaker compactness condition on another map S with $S(x) \subset T(x)$. This enables one to treat transfer-closed maps whose closures are KKM maps but do not satisfy the compactness conditions.

As applications of our new intersection theorems, we deduce new results on the existence of maximal and greatest elements for strict and weak relations arising in mathematical economics and we obtain new minimax inequalities which improve known results in [4].

Research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

0898-1221/04/$ - see front matter Typeset by $\mathcal{AMS}$-TEX
doi:10.1016/j.camwa.2004.03.003

2. A NEW INTERSECTION THEOREM

Let Y be a nonempty convex subset in a Hausdorff topological vector space E and X a nonempty subset of Y. We denote by 2^Y, the family of all subsets of Y, and by $\bar{B}$ and $\operatorname{co} B$, the relative closure of a subset B of Y and the convex hull of B, respectively. Let $G : X \to 2^Y$ be a multivalued map. We define G^{-1}, $G^* : Y \to 2^X$ and $G^c : X \to 2^Y$ by $G^{-1}(y) = \{x \in X : y \in G(x)\}$, $G^*(y) = X \setminus G^{-1}(y)$ and $G^c(x) = Y \setminus G(x)$. Some properties of the above maps can be found in Lemma 3.2 in [5] (also see [6, Lemma 2.1]). We shall apply those properties directly. We need to define a map $(\operatorname{co} G^*) : Y \to 2^{\operatorname{co} X}$ by $(\operatorname{co} G^*)(y) = \operatorname{co} G^*(y)$. Then, $(\operatorname{co} G^*)^*$ maps $\operatorname{co} X$ to 2^Y.

Recall that $G : X \to 2^Y$ is called a KKM map if for $\{x_1, \dots, x_n\} \subset X$,

$$\operatorname{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i).$$

The following new result provides necessary and sufficient conditions for a map to be a KKM map.

THEOREM 2.1. *The following conditions are equivalent.*

(i) $G : X \to 2^Y$ *is a* KKM *map.*
(ii) $x \notin (\operatorname{co} G^*)(x)$, *for each* $x \in \operatorname{co} X$.
(iii) $(\operatorname{co} G^*)^* : \operatorname{co} X \to 2^Y$ *is a* KKM *map.*

PROOF. First, we assume that (i) holds. If (ii) were false, then, $x \in (\operatorname{co} G^*)(x)$, for some $x \in \operatorname{co} X$. This implies that there exist $\{x_1, \dots, x_m\} \subset G^*(x)$, such that $x \in \operatorname{co}\{x_1, \dots, x_m\}$ and $x \notin \bigcup_{i=1}^{m} G(x_i)$, which contradicts (i). Next, we assume that (ii) holds. If (iii) were false, then, there exist $\{x_1, \dots, x_n\} \subset \operatorname{co} X$ and $x \in \operatorname{co}\{x_1, \dots, x_n\}$, such that $x \notin \bigcup_{i=1}^{n} (\operatorname{co} G^*)^*(x_i)$. This implies $x_i \in (\operatorname{co} G^*)(x)$ and we have $x \in (\operatorname{co} G^*)(x)$, a contradiction. Finally, (iii) implies (i) since $(\operatorname{co} G^*)^*(x) \subset G(x)$, for $x \in X$.

The following result is a generalization of Lemma 5.1 in [7]. Its proof is similar to that of Lemma 5.1 in [7] and thus, is omitted.

LEMMA 2.1. *Assume that* $G : Y \to 2^X$ *satisfies that* $G^{-1}(x)$ *is open in* Y, *for* $x \in X$. *Then,* $(\operatorname{co} G)^{-1}(x)$ *is open in* Y, *for* $x \in \operatorname{co} X$.

Recall that $T : X \to 2^Y$ is said to be transfer-closed if, for $x \in X$ and $y \in T^c(x)$, there exists $x_1 \in X$, such that $y \notin \bar{T}(x_1)$ (see [3]). If $T(x)$ is closed in Y, for $x \in X$, then, T is transfer-closed. Moreover, T is transfer-closed if and only if $\bigcap_{x \in X} \bar{T}(x) = \bigcap_{x \in X} T(x)$ (see [6, Lemma 2.3]).

The following result can be easily obtained by using Theorem 4.1 in [2].

LEMMA 2.2. *Assume that* $G : X \to 2^Y$ *is a* KKM *map and* $G(x)$ *is closed in* Y, *for* $x \in X$. *If* $\operatorname{co} X$ *is not compact, assume that there exist a nonempty compact convex subset* X_0 *of* X *and a nonempty compact subset* D *of* Y, *such that* $\bigcap_{x \in X_0} G(x) \subset D$. *Then,* $\bigcap_{x \in X} G(x) \neq \emptyset$.

Now, we are in a position to give our intersection theorem.

THEOREM 2.2. *Assume that* $S, T : X \to 2^Y$ *satisfy the following conditions.*

(H_1) $S(x) \subset T(x)$, *for each* $x \in X$.
(H_2) $\bar{S}$ *is a* KKM *map.*
(H_3) T *is transfer-closed.*
(H_4) *If* $\operatorname{co} X$ *is not compact, assume that there exist a nonempty compact convex subset* X_0 *of* $\operatorname{co} X$ *and a nonempty compact subset* D *of* Y, *such that*

$$\bigcap_{x \in X_0} \left(\operatorname{co}(\bar{S})^*\right)^*(x) \subset D.$$

Then, $\bigcap_{x \in X} T(x) \neq \emptyset$.

PROOF. We define a map $G : \operatorname{co} X \to 2^Y$ by $G(x) = (\operatorname{co}(\bar{S})^*)^*(x)$. By ($H_2$) and Theorem 2.1 (iii), G is a KKM map. Since $((\bar{S})^*)^{-1}(x) = (\bar{S})^c(x)$ is open in Y, for $x \in X$, it follows from Lemma 2.1 that $(\operatorname{co}(\bar{S})^*)^{-1}(x)$ is open in Y and thus, $G(x)$ is closed in Y, for $x \in \operatorname{co} X$. By ($H_4$) and Lemma 2.2, we have $\bigcap_{x \in \operatorname{co} X} G(x) \neq \emptyset$. Since $G(x) \subset \bar{S}(x)$, for $x \in X$, $\bigcap_{x \in X} \bar{S}(x) \neq \emptyset$. By ($H_1$) and ($H_3$), we have $\bigcap_{x \in X} T(x) = \bigcap_{x \in X} \bar{T}(x) \neq \emptyset$.

REMARK 2.1. If $S : X \to 2^Y$ is a KKM map, so is $\bar{S}$. However, the converse is false. For example, $S : [0,1] \to 2^{[0,1]}$ defined by $S(x) = \{0\}$ if $x = 0$ and $S(x) = [0, x)$ if $x \in (0,1]$ is not a KKM map but $\bar{S}$ is. Therefore, the map T in Theorem 2.2 needn't be a KKM map although $\bar{T}$ is a KKM map. We refer to Theorem 4.4 in [4] for an intersection theorem where the maps need not be KKM maps.

The following example shows that Theorem 2.2 is a proper generalization of Lemma 2.2 and Theorems 2 and 3 in [3].

EXAMPLE 2.1. Let $X = [0, \infty)$. We define two maps $S, T : X \to 2^X$ by

$$S(x) = \begin{cases} [0, 1 + x), & \text{if } x \in [0,1], \\ X, & \text{if } x > 1, \end{cases} \qquad \text{and} \qquad T(x) = \begin{cases} X \setminus (1,3), & \text{if } x = 0, \\ X \setminus [1 + x, 3), & \text{if } x \in (0,1], \\ X, & \text{if } x > 1. \end{cases}$$

Note that S satisfies (H_4) of Theorem 2.2 with $X_0 = \{0\}$ and $D = [0,1]$. It follows from Theorem 2.2 that $\bigcap_{x \in X} T(x) \neq \emptyset$. Since $\bigcap_{x \in X_0} \bar{T}(x)$ is not compact, for any subset $X_0 \subset X$, Lemma 2.2 and Theorems 2 and 3 in [3] can not be applied to T directly.

3. APPLICATIONS TO THE EXISTENCE OF MAXIMAL AND GREATEST ELEMENTS

Recall that a binary relation $\Re$ on $X \times Y$ is a subset of $X \times Y$. If $(x, x) \notin \Re$, for $x \in X$, then, $\Re$ is said to be a strict relation, denoted by $>$. If $(x, y) \in \Re$, we write $x > y$ and read $x > y$ as "x is (strictly) preferred to y". If $(x, y) \notin \Re$, we write $x \not> y$. We say X has a maximal element y in Y relative to $>$, if $y \in Y$ and $x \not> y$, for all $x \in X$. We define the strictly upper contour set of $y \in Y$ by $G(y) = \{x \in X : x > y\}$. It is obvious that X has a maximal element in Y relative to $>$, if and only if there exists $y_0 \in Y$, such that $G(y_0) = \emptyset$. The strict relations have been studied for example, in [3,5,7–9].

Recall that $G : Y \to 2^X$ is said to have the local intersection property if there exists an open neighborhood $N(y)$ of y, such that $\bigcap_{z \in N(y)} G(z) \neq \emptyset$, whenever $G(y) \neq \emptyset$ (see [6,10]). If $G : Y \to 2^X$ satisfies $G^{-1}(x)$ is open in Y for $x \in X$, then, G has the local intersection property. It is known that $G : Y \to 2^X$ has the local intersection property if and only if $G^* : X \to 2^Y$ is transfer-closed (see [6, Theorem 2.2]).

THEOREM 3.1. *Assume that $F, G : Y \to 2^X$ satisfy the following conditions.*

(i) *$G(y) \subset F(y)$, for $y \in Y$.*
(ii) *$\bar{F}^*$ is a KKM map.*
(iii) *G has the local intersection property.*
(iv) *If $\operatorname{co} X$ is not compact, assume that there exist a nonempty compact convex subset X_0 of $\operatorname{co} X$ and a nonempty compact subset D of Y, such that, for each $y \in Y \setminus D$,*

$$X_0 \cap \operatorname{co}(\bar{F}^*)^*(y) \neq \emptyset.$$

Then, there exists $y_0 \in Y$, such that $G(y_0) = \emptyset$.

PROOF. We define $S, T : X \to 2^Y$ by $S(x) = F^*(x)$ and $T(x) = G^*(x)$. By Theorem 2.2, we have $\bigcap_{x \in X} G^*(x) = \bigcap_{x \in X} T(x) \neq \emptyset$. Let $y_0 \in \bigcap_{x \in X} G^*(x)$. Then, we have $G(y_0) = \emptyset$. ∎

In Theorem 3.1, if $F = \operatorname{co} G$, we have the following.

COROLLARY 3.1. *Assume that $G : Y \to 2^X$ satisfies (iii) and (iv) of Theorem 3.1 with $F = \operatorname{co} G$ and $x \notin (\operatorname{co} G)(x)$, for $x \in \operatorname{co} X$. Then, there exists $y_0 \in Y$, such that $G(y_0) = \emptyset$.*

REMARK 3.1. Corollary 3.1 generalizes Theorem 2 in [11], where $X = Y$ is closed and $S^{-1}(x)$ is open, for $x \in X$.

Recall that a binary relation A on $X \times Y$ is said to be a weak relation, denoted by $\leq$, if $(x,x) \in A$, for $x \in X$. If $(x,y) \in A$, we write $x \leq y$ and read $x \leq y$ as "x is at most as good as y". We say X has a greatest element y in Y relative to $\leq$, if $y \in Y$ and $x \leq y$, for all $x \in X$. X has a greatest element in Y relative to $\leq$ if and only if there exists $y \in Y$, such that $X \times \{y\} \subset A$. Weak relations have been widely used for example, in [3,5,9].

THEOREM 3.2. *Assume that $A, B \subset X \times Y$ satisfy the following conditions.*

(i) *$B \subset A$.*
(ii) *$y \notin \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : (x,z) \in B\}}\right\}$, for $y \in \operatorname{co} X$.*
(iii) *If $(x,y) \notin A$, there exist an open neighborhood N_y of y in Y and $x_1 \in X$, such that $(x_1, z) \notin A$, for $z \in N_y$.*
(iv) *If $\operatorname{co} X$ is not compact, assume that there exist a nonempty compact convex subset X_0 of $\operatorname{co} X$ and a nonempty compact subset D of Y, such that, for $y \in Y \setminus D$,*

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : (x,z) \in B\}}\right\} \neq \emptyset.$$

Then, there exists $y_0 \in Y$, such that $X \times \{y_0\} \subset A$.

PROOF. We define two maps $S, T : X \to 2^Y$ by

$$S(x) = \{y \in Y : (x,y) \in B\} \qquad \text{and} \qquad T(x) = \{y \in Y : (x,y) \in A\}.$$

By Theorem 2.2, we have $\bigcap_{x\in X} T(x) \neq \emptyset$. Let $y_0 \in \bigcap_{x \in X} T(x)$. Then, $X \times \{y_0\} \subset A$. ∎

REMARK 3.2. Theorem 3.2 with $A = B$ generalizes Lemma 4 in [1], where X is compact and A is closed in $X \times X$.

4. APPLICATIONS TO MINIMAX INEQUALITIES

Let $\lambda \in \mathbb{R}$. Recall that $f : X \times Y \to \mathbb{R}$ is said to be λ-transfer-lower-semicontinuous on Y if there exist an open neighborhood $N(y)$ of y and $x_1 \in X$, such that $f(x_1, z) > \lambda$, for $z \in N(y)$ whenever $f(x,y) > \lambda$.

By Theorem 2.2, we obtain the following new variational inequality for functions defined on $X \times Y$.

THEOREM 4.1. *Let $\lambda \in \mathbb{R}$. Assume that $f, g : X \times Y \to \mathbb{R}$ satisfy the following conditions.*

(i) *$f(x,y) \leq g(x,y)$, for $(x,y) \in X \times Y$.*
(ii) *$x \notin \operatorname{co}\{x \in X : y \notin \overline{\{z \in Y : g(x,z) \leq \lambda\}}\}$.*
(iii) *f is λ-transfer-lower-semicontinuous on Y.*
(iv) *If $\operatorname{co} X$ is not compact, assume that there exist a nonempty compact convex subset X_0 of $\operatorname{co} X$ and a nonempty compact subset D of Y, such that, for $y \in Y \setminus D$,*

$$X_0 \cap \operatorname{co}\{x \in X : y \notin \overline{\{z \in Y : g(x,z) \leq \lambda\}}\} \neq \emptyset.$$

Then, there exists $y_0 \in X$, such that $f(x, y_0) \leq \lambda$, for $x \in X$.

PROOF. We define two maps $S, T : X \to 2^Y$ by

$$S(x) = \{y \in Y : g(x,y) \leq \lambda\} \qquad \text{and} \qquad T(x) = \{y \in Y : f(x,y) \leq \lambda\}.$$

By (iii) and Proposition 3.1 in [4], T is transfer-closed. The result follows from Theorem 2.2. ∎

REMARK 4.1. When $f = g$, Theorem 4.1 improves Corollary 6.1 in [4], where f satisfies extra conditions. Note that Theorem 4.1 requires $X \subset Y \subset E$, while in Theorem 6.1 in [4], X and Y may belong to different Hausdorff topological vector spaces.

By Theorem 4.1, we obtain the following new inequalities which improve Theorem 6.2 and Corollary 6.2 in [4].

THEOREM 4.2. *Suppose $f, g : X \times Y \to \mathbb{R}$ satisfy the following conditions.*

(i) $\lambda_g \in [-\infty, \infty)$, *where* $\lambda_g = \sup_{x\in X} \inf_{y\in Y} g(x, y)$.
(ii) $f(x, y) \leq g(x, y)$, *for* $(x, y) \in X \times Y$.
(iii) f *is λ-transfer-lower-semicontinuous on* Y, *for* $\lambda > \lambda_g$.
(iv) $x \notin \operatorname{co}\{x \in X : y \notin \overline{\{z \in Y : g(x, z) \leq \lambda\}}\}$, *for* $\lambda > \lambda_g$.
(v) *If* $\operatorname{co} X$ *is not compact, assume that there exist* $\lambda_1 \in (\lambda_g, \infty)$, *a nonempty compact convex subset* X_0 *of* $\operatorname{co} X$ *and a nonempty compact subset* D *of* Y, *such that, for* $y \in Y \setminus D$,

$$X_0 \cap \operatorname{co}\{x \in X : y \notin \overline{\{z \in Y : g(x, z) \leq \lambda_1\}}\} \neq \emptyset.$$

Then, the following inequalities hold:

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} \inf_{y\in Y} g(x, y) \leq \inf_{y\in Y} \sup_{x\in X} g(x, y).$$

PROOF. Let $\lambda \in (\lambda_g, \lambda_1]$. Then, it is easy to verify that f, g, and λ satisfy all the conditions of Theorem 4.1. Hence, there exists $y_0 \in Y$, such that $f(x, y_0) \leq \lambda$, for all $x \in X$. This implies $\inf_{y\in Y} \sup_{x\in X} f(x, y) \leq \lambda$, for $\lambda \in (\lambda_g, \lambda_1)$ and $\inf_{y\in Y} \sup_{x\in X} f(x, y) \leq \lambda_g$. This implies that the first inequality holds. It is clear that the second inequality holds. ∎

By Theorem 4.2 and Theorem 3.1 in [4], we obtain the following.

COROLLARY 4.1. *Let X and Y be two nonempty compact convex subsets of E with $X \subset Y$. Assume that $f : X \times Y \to \mathbb{R}$ satisfies (i), (iii), and (iv) of Theorem 4.2 with $g = f$. Then,* $\min_{y\in Y} \sup_{x\in X} f(x, y) = \max_{x\in X} \inf_{y\in Y} f(x, y)$.

Corollary 4.1 generalizes Corollary 6.3 in [4], Theorem 3.4 in [12], Theorem 5 in [13], and Theorem 16 in [14].

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